From Infinity to Irrationality and Back Again:

An investigation of the classical foundations of the calculus.

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Spring 2004

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INTRODUCTION

Mathematics has been around ever since man first learned to count on his fingers. The Egyptian and Babylonian civilizations each had their own developments in numbering systems and calculation techniques. It was the Greeks, though, who made the change that has shaped the mathematical world ever since. They developed the discipline of rigorous proof. For 2500 years, that has been the guiding principle in the world of mathematics. Greek mathematical texts contain the first examples of logical proof. These are the first mathematical writings that have completely retained their value to the present day.

This paper will serve as an introduction to Greek mathematics. I will use the development of calculus to focus my study of the subject. This particular focus is intriguing because calculus is one of the parts of mathematics least associated with the Greeks. Traditionally, Newton and Leibniz 'created' calculus somewhere around 1700 C.E. What they really did was begin to codify ideas whose seeds had been planted thousands of years earlier. I will trace these ideas to their roots in classical antiquity.

HISTORICAL OVERVIEW

Greeks began studying math in an organized fashion sometime around 600 B.C.E. Schools developed in two locations: Asia minor and southern Italy. The school in Asia minor was founded by Thales, who is usually credited with being the first natural philosopher. There are no great results that came from this school. However, several basic theorems are attributed to Thales and his school. Proclus reports that Thales discovered that the circle is bisected by its diameter (157.10 – 13), the base angles in an isosceles triangle are equal (250.22 – 251.2), and the intersection of two straight lines produces 2 sets of equal angles (299.1 – 5). Proclus also tells us that Thales somehow used the equality of triangles to determine the distance of a ship from shore (352.14 – 18). Many modern scholars have proposed methods he might have used. Heath's theory is that he stood on a tower and measured the angle of declination to the ship, then turned around and found an object on the mainland at the same angle of declination. This created 2 congruent triangles where the distance to the ship and to the fixed point on land was equivalent (The Thirteen Elements of Euclid's Elements, vol. i, 305).

The Pythagorean school arose in Southern Italy. According to Anatolius (cited by Heron) they were the first to coin the term "μαθηματα," (160.8 – 162.2). Brunes argues that the greatest achievement of Pythagoras was to take mathematics out of the temple. Pythagoras spent a great deal of time learning cult mysteries in Egypt. According to Brunes, computational methods and geometric theorems were developed in religious contexts from very early times. The radical change Pythagoras wrought was the de-mystification of math. Pythagoras did teach a philosophy centered around the mystical connection numbers had to the universe. However, he did not ask his followers to accept this connection on faith, instead they investigated numbers and shapes to show by proof the intricacies of mathematics. Through this investigation, the followers of Pythagoras laid the foundation for the logical pursuit of mathematics in the centuries to follow.

In the 5^{th} and 4^{th} centuries, Athens was the world's intellectual center. Accordingly, mathematical scholarship was focused there. Plato's school was very active in the development of mathematics. Specifically, they were interested in problems dealing with infinity and other abstract mathematical concepts (Boyer). Competing with this group was Aristotle and his followers. Aristotle defined a potential infinite and denied the actual, spatial infinite (Physics Γ

6, 206a 9 – 16). This view of the infinite discouraged active study in that direction. Because Aristotelian influence was dominant in mathematics until the Renaissance, this significantly impeded the progress of calculus (Boyer).

There were many individuals from both schools whose work was notable, some of whom will be mentioned below. All of their work was collected and categorized by the next character in the drama of math, Euclid.

EUCLID

Euclid was a Greek mathematician who worked around 300 B.C.E (Heath, Thirteen Elements of Euclid's Elements, vol. i, 2). Very little is actually known about his life, though there are countless legends surrounding him. Arab commentators started the tradition that his birthplace was Tyre in Asia minor, but this is dubious because it is usually accompanied by outrageous statements which further easternize his work. For instance, Thus Nasiraddin, the Arabic translator of the *Elements*, gave Euclid the surname of Thusinus, making him out to be a relative (4). Apparently this was a common practice of Arab translators. They described Pythagoras as a pupil of Salomo and they made Aristotle an Egyptian. Medieval Christian scholars thought Euclid's birthplace was Megara. This however, was a result of confusion between two people, because there was a 4th century B.C.E. philosopher by the name of Euclid of Megara.

One of the stories told about Euclid's life comes from Stobaeus. It is one of countless tails about Euclid which has no basis in fact. It reads as follows: "some one who had begun to read geometry with Euclid, when he had learnt the first theorem, asked Euclid, 'But what shall I get by learning these things?' Euclid called his slave and said 'Give him threepence, since he must needs make profit out of what he learns,'" (Stobaeus, Extracts ii. 31.114 trans. Thomas).

It is most likely that Euclid received his education in Athens at the hands of Plato's pupils (Heath, Thirteen Elements of Euclid's Elements, vol. i, 2). His magnum opus is *The Elements*, a compendium of the mathematical knowledge of his day. Most of the material in it was not created by Euclid but collected there by him. We are lucky that Euclid undertook this project because most of the works he references are lost.

Most of our reliable information about Euclid and the sources he drew upon comes from Proclus, his primary commentator. Proclus lived from 410 to 480 C.E. He was a neo-Platonist who taught mathematics in Athens. He wrote many commentaries on Platonic dialogues, and one commentary on Book I of Euclid's *Elements* (Heath, Thirteen Elements of Euclid's Elements, vol. i, 29). Although he was separated by a vast gulf of time from the pre-Euclidean geometers whom he discusses, his information is usually the most reliable. This is because he writes honestly about his own sources and describes how he comes to his conclusions. For instance, to place Euclid in time, he looks at references to him by Archimedes, and other sources who connect Euclid to a Ptolomey. Since Euclid must then have lived before Archimedes, and during the reign of a Ptolomey, it must have been during the reign of Ptolomey I, who is the only one to reign before Archimedes was writing (Heath, Thirteen Elements of Euclid's Elements, vol. i, 1).

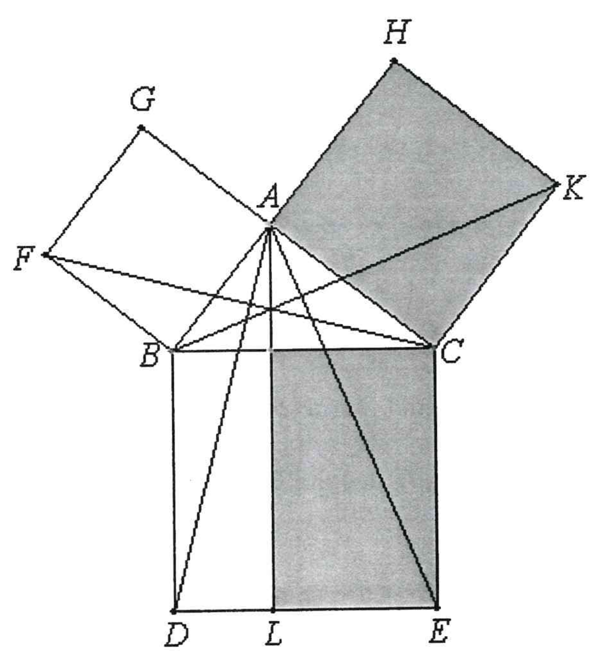
PYTHAGOREAN CONTRIBUTIONS

The most famous product of Pythagorean study is, of course, the Pythagorean Theorem. It relates the lengths of the legs of a right triangle to the length of the hypotenuse. Stated in modern mathematical terms, it reads $a^2 + b^2 = c^2$ where a and b are the lengths of the legs and c is the length of the hypotenuse. How would this simple formula be understood by a classical mathematician? To answer this question, we must first look at the divisions of mathematics in the classical mind.

When Martianus Capella created his model for balanced education, *De nuptiis Mercurii et Philologiae*, he included two books on mathematics, Geometry and Arithmetic. This division was common across the entire classical world – with the exception of the Pythagorean order. The order was based on a religion that bound the two realms of math together into the first grand unified theory to explain the physical universe through mathematics (Boyer). This theorem could be an attempt to bridge the gap between the realms of numbers and shapes. We do not know how the Pythagoreans proved this theorem, but we do have several later methods. The most famous proof is that offered by Euclid in Book I of *The Elements*.

EUCLID'S PROOF OF THE PYTHAGOREAN THEOREM

Let us examine the text of Euclid's proof in detail. Basically the he shows that the yellow and blue squares are equal to the yellow and blue areas in the largest square. As one might expect, a discipline as formal as mathematical proof lends itself to structured, rigid language. As he states the proposition he uses ring composition (ABCCBA form) to convey his point clearly. While direct translation of this



form makes for a very confusing and ambiguous English sentence, the Greek is very specific and clear. This unfamiliar syntax is one of the challenges of reading Euclid or any other Greek mathematical text, even for a mathematician.

A second difficulty common to all mathematical texts is their specific vocabulary. The Greeks had a system for talking about lines, points, and angles which is subtly different from ours. We refer to the 'legs' of a right triangle, where Greeks would call them 'ribs' (πλευράς). In fact, that term extends to the sides of all polygons. We call the side of the right triangle opposite the right angle the hypotenuse. To talk about this side, Euclid must use a participial phrase like, ἡ τὴν ὀρθὴν γωνίαν ὑποτείνουσα πλευρά. Clearly, our word 'hypotenuse' comes from ὑποτείνειν, but we are able to use it without the awkward phrase.

Another difference in expression between Euclid and modern mathematicians is apparent from the statement of the Pythagorean theorem. We state the theorem simply as $a^2 + b^2 = c^2$. Euclid cannot use this algebraic simplification for several reasons. First, he does not have the equal sign or addition symbol, since symbolic algebra of this nature was a later Islamic invention. Euclid was certainly certainly aware of multiplication and that the area of a square could be found by multiplying the length of its side by itself. In fact, in Book VII (devoted to what we call number theory), he defines a 'square number' (τετράγωνος αριθῆος) as a number which is a smaller number multiplied by itself. To Euclid, and the classical world, geometry is the purest form of mathematics. I suspect this is why he chooses to prove this theorem by a strictly geometric method.

Euclid begins each proof with a statement of what he is trying to prove. Here, he states, "In right triangles, the square from the rib stretching under the right angle is equal to the squares from the ribs surrounding the right angle."[1] It is interesting to note that in Euclid's statement, there is no mention of addition. This is a convention in Euclid and elsewhere. Addition is the natural way to combine two areas, and therefore not specified.

The format which the proof takes is another Pythagorean innovation. It is called the application of areas. It could legitimately be called the first step in the direction of modern calculus. It was the first method of comparing dissimilar shapes. They broke the largest shape down into components for comparison to other shapes. In this way, they could begin to show if shapes were larger, smaller, or equal in size. The Pythagoreans actually refined this to the point where they could reduce any straight-sided, closed figure to a triangle, and thus to a square (Baron, 28). This was the Greek way of avoiding the connection between numbers and shapes. Now we can simplify the whole process by comparing the numerical values for areas of shapes.

THE FLAW IN THE PYTHAGOREAN WORLD

This theorem proved to be the downfall of the Pythagorean philosophy because it is evidence for the existence of what we call irrational numbers, called "unspeakable" (ἄλογος) by the Pythagoreans (Danzig). Their system was based on integers, whole numbers like 1, 2, and 3. They did not have fractions as we think of them today, but only as the ratio between two whole numbers. For instance, the relationship between 15 and 5 would be 1/3. In this way, the Pythagoreans had what we call the rational numbers, which include the integers and all fractions made with them. However, when we consider a right triangle with both sides having length 1, we quickly find that the hypotenuse is $\sqrt{2}$.

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[1] Unless otherwise specified, all translations are my own. Heath translates this passage thus: "In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle."

The followers of Pythagoras were initially able to ignore this by simply assuming it was a rational relationship in terms too large for them to calculate (Danzig, 100). Eventually, though someone proved that the diagonal of a square is truly incommensurable with its side. This was such a disturbing discovery in the Pythagorean world that, according to Proclus, the first Pythagoreans to speak of it to outsiders died at sea as punishment (Danzig, 101). The proof is a very elegant number theory exercise that was traditionally included in Book X of Euclid's *Elements*. This proof was relegated to an appendix by August and Heibert, editors of the two most current Greek editions of Euclid because it appears to have been added later as an interpolation. Regardless, it is a fundamental proof taught in every basic analysis class to this day.

PROOF OF THE IRRATIONALITY OF $\sqrt{2}$

Despite its apparent interpolation, this is still a fundamental proof for the study of calculus, and it was known to the Greeks. So, for our purposes, whether or not Euclid actually included it in the *Elements* is immaterial.

Due to an oddity in Euclid's definition of ἄλογος (see the Glossary), he is not proving that that $\sqrt{2}$ is irrational (by his definition, not ours), but merely incommensurable with a given quantity (which is our definition of irrational). Instead he is proving that the two quantities are ἀσύμμετρος, or incommensurable. This means that there is no common into unit which both lengths can be broken down to a finite number.

Euclid sets this proof up by examining a square with its diagonal drawn in. The object of the proof is to show that the diagonal of the square is incommensurable with the side of the square. The method is incredibly elegant, and never fails to amaze a student of mathematics. The method of the proof is to show that for there to be a common measure between diagonal and side, one of the side must be both even and odd.

This proof follows another Pythagorean technique, the *reduction ad absurdum*, or proof by contradiction, a common proof tactic to this day. In the proof, the author creates the square and its diagonal, and then creates a second a pair of segments in the same ratio to each other as the square's side and diagonal. This is an interesting step because it is generally omitted today. The author, however, needs to create this intermediate step so that he can modify the new segments geometrically, which he cannot do to the originals because they are part of his square. Anything he proves about the created segments applies to the side and diagonal because he created the new segments in their image.

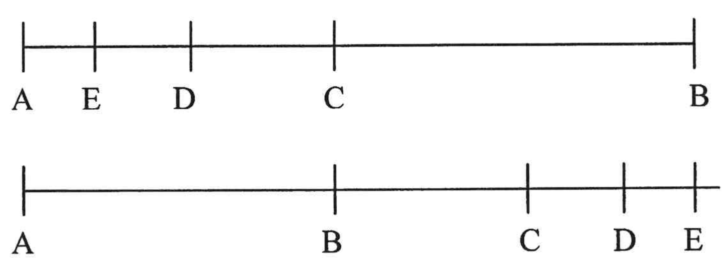
In this proof, as in the Pythagorean Theorem proof, the author does not use the algebraic idea of squaring, but rather the geometric. It is an interesting stylistic point that, in this proof, the author omits the word for square. For instance, "το απο της ΓΑ," suffices to mean "the square upon the side gamma-alpha." This feature is consistent throughout the proof. The author also goes into some detail to establish the legitimacy of comparisons between the created segments and the side and diagonal. However, he only does this once, expecting the reader to repeat his steps the second and third times they occur.

PARADOXES OF ZENO

No discussion of calculus in antiquity is complete without mention of the paradoxes of Zeno. They were a set of problems that encouraged debate and thought about infinity more than any other single force.

Zeno was a philosopher and mathematician working about 460 B.C.E. (Baron, 21). He presented four paradoxes to the world, each having to do with the concept of infinity, the infinitesimal, and movement. There is much speculation as to the motivation behind the paradoxes, but nobody can be sure whether Zeno was proposing them as a mere logical puzzle or he was proposing them as a genuine mathematical paradox. In large part, this is because the classical writings on the subject tend to be very bitter and recalcitrant (Baron, 21).

In two of the paradoxes, *Dichotomy* and *Achilles*, Zeno argues that if time and space are infinitely divisible, motion is impossible. Basically, to get from point A to point B (in the upper diagram), we must first traverse the distance between A and C (a point midway between A and B). To get from A to C, one must pass through D (midway between A and C). These statements repeat themselves ad infinitum. Therefore, to get to any point, we must first get halfway there, and since there is always another closer point to hit on the way, we never get anywhere. This is Zeno's *Dichotomy* (Baron, 21).



Achilles is the same paradox seen in the other direction. If Achilles is chasing a tortoise, and the tortoise starts at B, while Achilles starts at A (in the lower diagram), Achilles will never catch the tortoise. This is because when Achilles reaches B, the tortoise is already at C, and when Achilles reaches C, it is at D, etc (Baron, 21).

Zeno's other two paradoxes begin with the assumption that time and space are not infinitely divisible. Instead, there is such a thing as a smallest unit of time, an instant, and a smallest unit of space, a point. If an arrow is flying through the air, at a given instant, it must occupy a certain point. Since it can only be in one place per instant, it cannot move in that instant. Since the same logic holds for all instants, the arrow can never move (Baron 22).

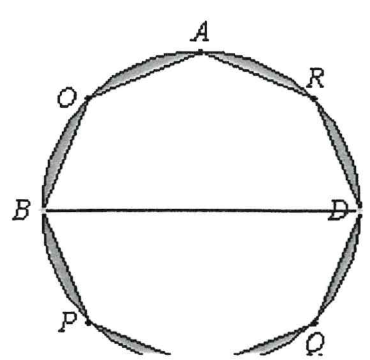
The fourth paradox, the *Stadium*, is by far the least easily understood. The main source for its description is its refutation in Aristotle's *Physics*. The description of the paradox is extremely antagonistic and confusing, which serves to make Aristotle's counter-argument even more appealing. There are two main scholarly explanations, both of which agree that it is a valid paradox in which half of a given time is equal to the whole time (Thomas, vol. i, 374).

These paradoxes have been the subject of much discussion in the mathematical world to this day. The *Stadium* has even been compared to the special theory of relativity, and does indeed bear a certain resemblance (http://www.mathpages.com/rr/s3-07/3-07.htm).

SQUARING THE CIRCLE:

THE METHOD OF EXHAUSTION

The next step in the direction of calculus is traditionally attributed to Eudoxus. This pre-Euclidean geometer of the platonic school devised a way to create a polygon with area equal to that of a circle. This is another problem involving irrational numbers, because the area of a circle is πr^2, and π is an irrational number (which cannot be represented accurately by a fraction).



Therefore, to make a polygon of equal area, the polygon must have a side of irrational length. Eudoxus' method involved inscribing a regular polygon inside a circle and successively doubling the number of sides it had. By this "method of exhaustion," one could create a polygon with area "as near as desired" to the area of the circle. This concept anticipates our modern concept of the limit, upon which calculus is based.

THE DIFFICULTY OF ORIGINS

The process described above is given in Book XII of Euclid's *Elements*. This theorem is said to have been proved by Hippocrates, but is most likely the work of Eudoxus (Heath). About 50 years after the *Elements* was written, Archimedes attributed two very similar theorems from Book XII to Eudoxus, and based on that evidence, Eudoxus is credited with the method of exhaustion. The canon of classical math texts is full of confusing attributions and references to lost works. This method is particularly confused in its origins.

Antiphon is the first person attributed with the creation of this method. He was a sophist contemporary to Socrates (middle 5th century B.C.E.). Aristotle in the *Physics* mentions that Antiphon's method for squaring the circle is sound (as opposed to another unnamed method) and therefore not to be refuted. Commentators on Aristotle have expanded on this passage. Themistius, a 4th century C.E. Greek commentator who also translated Aristotle into Arabic, says that the unsound method Aristotle mentions was that of Hippocrates of Chios. He also describes Antiphon's method more fully as inscribing a series of triangles, which is logically equivalent to the method in Euclid XII, 2. This version of Antiphon's method is considered to be most accurate because it is the earliest description we have (Heath). Simplicius, a 5th century comentator, objects to Antiphon's method because it would never fully cover the circle. This is true because the Greek mind did not conceive of this division being carried out to an infinite degree as we would today.

The next Greek credited with this method of squaring the circle is Bryson, a geometer one generation after Antiphon (late 5^{th} to early 4^{th} century B.C.E.). Alexander of Aphrodisias, an early 3^{rd} century C.E. commentator on Aristotle's *Sophistic Refutations*, describes Bryson's method. It was similar to the method described in Euclid XII, 2 but instead approached the circle with polygons circumscribed and inscribed.

For none of these methods do we have the original statement and proof. Only in the case of Euxoxus' method do we have any proof at all. The others only remain to us in scraps of commentary written 700 - 1000 years after the fact. In fact, the ambiguity of these methods has led some to characterize them as "a well-established algorithm of the differential calculus," (Boyer). The great leap which Newton and Leibniz made was to provide an explicit algorithm to accomplish what Eudoxus and others had proposed was possible.

CONCLUSION

The world of Greek mathematics is rich in complexity and steeped in tradition. This depth is certainly something that has been worthwhile for me to study over the course of this year. I feel that I have found made some connections at least in my own mind which link my two disciplines: classics and math. Looking back at some of the two thousand year old ideas presented in this paper, which I did not encounter until reaching college, I appreciate the classical heritage even better now than I did one year ago.

Appendix One – Glossary

This glossary contains terms from Greek mathematics. This is a sampling of vocabulary which I ran across in my studies. I chose the words by virtue of their interesting etymologies and importance to calculus. All definitions are taken from Liddell and Scott. Quotations were accessed either by the Perseus Project or the Loeb Classical Library. The Loeb compilation of mathematical works (edited by Ivor Thomas) was particularly useful.

ἄλογος

This word is λόγος with an alpha-privative. Thus it means the opposite of λόγος. As the opposite of the speech-based meanings of λόγος, it can mean *speechless* (Plato). As an opposite to reason/explanation, it can mean *unreasoning or unthinking*. This definition is extended to to refer to *unthinking animals* (Xenophon).

To counter the computation-based meanings of λόγος, ἄλογος can mean *without reckoning or counting*. Thucydides uses it to mean not *counted-upon or unexpected.* In mathematics it means that *two magnitudes are incommensurable*. If two lines are incommensurable, the ratio between them must be an irrational number (Democritus). Euclid defines ἄλογος in a subtly different way from most (see below).

Because of the two meanings of ἄλογος, Danzig is very perceptive when he translates the Pythagorean ἄλογον as the "unutterable."[2] Literally the text referred to irrational numbers, which were a great secret of the Pythagorean order.

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[2] Danzig, *Number, The Language of Science.*

Significant uses:

Democritus wrote an entire book entitled, "περὶ ἀλόγων γραμμῶν," (About Incommensurable Lines).

Plato – Laws: 696e – "ἀλόγου σιγῆς" – "Speechless silence," or temperance, is a virtue to be highly esteemed.

Xenophon – Hiero: 7.3 – "ἐν τοῖς ἀλόγοις ζῴοις ἐμφύεται" – Simonides tells Hiero that the love of honour "is not implanted in unthinking animals."

Thucydides – Histories: 6.46.2 – "καὶ ἀλογώτερα" – This is an interesting passage over which there is much disagreement. Everyone agrees that it has to do with an unexpected event, but scholars disagree over what exactly that event is, and for whom it was unexpected.

Euclid – Elements: Book 10, Definition 3:

> Τούτων ὑποκειμένων δείκνυται, ὅτι τῇ προτεθείσῃ εὐθείᾳ ὑπάρχουσιν εὐθεῖαι πλήθει ἄπειροι σύμμετροί τε καὶ ἀσύμμετροι αἱ μὲν μήκει μόνον, αἱ δὲ καὶ δυνάμει. καλείσθω οὖν ἡ μὲν προτεθεῖσα εὐθεῖα ῥητή, καὶ αἱ ταύτῃ σύμμετροι εἴτε μήκει καὶ δυνάμει εἴτε δυνάμει μόνον ῥηταί, αἱ δὲ ταύτῃ ἀσύμμετροι ἄλογοι καλείσθωσαν.
>
> With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in

> length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called rational, and those straight lines which are commensurable with it, whether in length and in square or in square only, rational, but those which are incommensurable with it irrational.[3]

This is a very interesting definition because it includes perfect square roots as rational numbers. Euclid uses σύμμετρος and ἀσύμμετρος to mean commensurable and incommensurable. He then defines rationality as an extension upon commensurability. This definition is reproduced in its entirety because it is essential to the study of calculus in antiquity.

ἀριθμός

This word has 13 main definitions and 6 subordinate meanings in Lidell, Scott, and Jones. The basic meaning is *number.* With an ordinal, it can refer to a *term in a sequence* or to a *numeral* (Euripides and Sextus Empiricus). With a genitive, it can mean *amount* (Xenophon). It was also frequently used to describe a person's *station in society or relative worth* (Homer). It came to be associated with *numbering or counting* (Pindar).

Diophantus, the great Greek Algebraist used it as his word for *abstract quantity.* In other scientific disciplines, it gained specialized meaning as well. In medicine, it was used to refer to *precise conditions under which an operation could be performed.* In Astronomy, it came to refer to *the number of degrees a celestial body had traversed in a given period of time* (or its speed).

Significant Uses:

Homer – Odyssey: 11.449 – "μετ' ἀνδρῶν ἵζει ἀριθμῷ" – Agamemnon says that Telemachus "takes his number among men."

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[3] Heath's translation.

Pindar – Nemean Odes: 2.23 - "μάσσων ἀριθμοῦ" - The victories of Timodemus at home are "past counting."

Xenophon – Anabasis: 2.2.6 – "ἀριθμὸς τῆς ὁδοῦ ἣν ἦλθον ἐξ Ἐφέσου" – "The *amount* of road they had taken from Ephesus was …"

Euripides – Ion: 1014 – "ὁ δεύτερος δ' ἀριθμὸς ὧν λέγεις" – "The second one of which you speak," this is ἀριθμὸς with an ordinal referring to a term in a sequence.

Sextus Empiricus – Adversus Mathematicos: 7.96 - "□̣□□□□̣□̣□□□□□□□" – "The fourth number," here meaning the number 4.

Diophantus – Definition 2 – He defines □□□□□□□ as "πλῆθος μονάδων ἀορίστων" or, "a multitude of uncertain unit numbers." We would say an uncertain multitude of units

ἄρτιος

This word means *complete, perfect, or fitted to its purpose.* By applying complete to a living creature, it came to mean *full-grown.* It also came to mean *prepared or ready* (Herodotus). As an adverb, it means *newly* (Sophocles).

In mathematics, it means that a number is *even.* This sheds an interesting light on the connections the ancients saw between numbers and religion. This topic is delved into in great detail in Martianus Capella's chapter on Arithmetic.

Significant Uses:

Homer – Iliad: 14.92 – "ἄρτια βάζειν" – Odysseus begs Agamemnon to "say things which are fitting" of a man in his station.

Herodotus – Histories: 9.27 – "ἄρτιοι πείθεσθαι" – The Athenians feel that the enemies appear to be "ready to be persuaded" to concede defeat.

Sophocles – Ajax: 678 – "ἐπίσταμαι γὰρ ἀρτίως ὅτι" – An angry Ajax "newly learns that" an enemy is ceases to be an enemy when it interferes with another alliance.

Plato – Protagoras: 356e – Socrates invites questions Protagoras about a hypothetical situation in which the choice between even and odd was to determine a man's life or death.

λόγος

This is another word who's entry stretches on forever in any Greek lexicon. Liddell and Scott list meanings varying from t*he Word of God, to an utterance, to computation*, and finally to *proportion*. It is this last meaning that is most important in the world of mathematics.

λόγος is the verbal noun of λέγω. It corresponds in meaning to two of the meanings of λέγω: *to count and to speak*. The speech-based definitions are not relevant to mathematics, and are not included here.

One main definition is a *computation or counting*. This usage was often associated with money, and was even used as a title for treasurer. It is also used to mean *an account or reckoning*. With elements of communication and calculation, this straddles the line between the speech and counting based definitions. Along with this meaning is the sense of esteem or value: *the sum total of something's worth* (Sophocles).

Another main computation-based meaning of λόγος is *relation, correspondence, or proportion* (Aeschylus, Plato). Grammarians used it to mean a *rule of language*. In mathematics, it is used to define a *proportion* (Euclid, Aristotle). Greek mathematicians could only define a length, area, volume or other magnitude in relationship to a second one (line A has length 4 times greater than line B).

Significant Uses:

Aeschylus – Seven against Thebes: 519 – "πρὸς λόγον τοῦ σήματος" – Eteocles predicts that the battle between mortals will go "according to the relationship between the signs" portrayed on their shields.

Sophocles – Oedipus at Colonus: 1225 – "μὴ φῦναι τὸν ἅπαντα νικᾷ λόγον" – The chorus laments about the woes of life, saying: "Not to be born exceeds every possible *account/sum total* (of life)."

Plato uses the phrase ἀνὰ λόγον to mean proportionally in several instances. In Phaedo 110d, the growing things of earth are proportionally beautiful. In Timaeus 37a, the three natures of the soul are divided and combined proportionally.

Aristotle – Metaphysics: 985 b32 – "τῶν ἁρμονιῶν ἐν ἀριθμοῖς ὁρῶντες τὰ πάθη καὶ τοὺς λόγους" – The Pythagoreans saw that the "properties and proportions of musical scales" had to do with numbers. Here the proportion is between the lengths of the strings stretched across one's lyre.

Euclid – Elements: Book 5, Definition 3 – "Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις" – This is Euclid's definition of λόγος: "A ratio is a sort of relation in respect of size between two magnitudes of the same kind."[4]

μονάς

This is the feminine form of the adjective μόνος, which has an extensive definition in Liddell and Scott. Basically, it means *alone or solitary*. The feminine form can be used substantively to mean a *solitary female*. However, it is more commonly used to mean *unit or monad*. Mathematically, this is the abstract concept of a single unit with no set length. To prove two things are commensurable, they must be able to broken into a finite number of pieces of this unit size.

An interesting connection with the Pythagoreans exists. Their mystic religious/philosophical system was based on numbers and elements. To them, this word was connected with the element of fire. Μονάς is the most basic mathematical unit, and it was the center of the Pythagorean universe (Plutarch).

[4] Heath's translation.

Significant Uses:

Euripides – Andromache: 855 – "μονάδ' ἔρημον" – Hermione has been abandoned on the shore in "desolate solitude" without even an oar.

Plutarch – Life of Numa: 11.1 – "και τουτο Έστιαν καλουσι και μοναδα" – Numa built the temple to Vesta in a circular shape to represent the shape of the universe, at the center of which the Pythagoreans place the element of fire, "and call it Vesta and Unit."

περισσός

This word means: *beyond the regular number or size, prodigous*. The word developed both positive and negative connotations. It could mean *extraordinary or remarkable* (Euripides) or *superfluous* (Xenophon).

In mathematics, this word is used to describe odd numbers. It can be used to describe the ratio between lines, areas, or volumes. This word, along with ἄρτιος are very important in the proof that $\sqrt{2}$ is incommensurable with a rational number.

Significant Uses:

Hesiod – Theogony: 599 – "περισσὰ δὲ δῶρα δέδωκεν" – Zeus gave "a huge number of gifts" to Styx.

Euripides – Hippolytus: 948 – "περισσὸς ὢν ἀνὴρ" – Theseus tells Hippolytus that he is a "man apart," who comes and goes with the gods.

Xenophon – Memorabilia: 3.6.6 – "δαπάνας … περιττὰς" – Socrates criticizes Glaucon's fiscal policies, asking if he will cut out "excess expenditures."

πλευρά

The root definition of this word is *rib*, and it is usually found in the plural. Often it is used in metonymy for the *side of a person or animal*. In mathematics, it came to mean the *side of a polygon or triangle*. From its usage as a side, it was expanded to several concepts the Greeks associated with sides. It came to mean a *factor of a number* because the Greeks implicitly associated multiplication with the geometric operation of finding the area of a rectangle. From factor it came to be used for only the factors of perfect squares or cubes: *square or cube roots*. In at least one case, the word came to stand for an *infinite line* (see Archimedes below).

Significant uses:

Homer – Iliad: 24.10 – "ἐπὶ πλευρὰς κατακείμενος" – Achilles is "lying on his side" as he grieves for Patroclus.

Herodotus – Histories: 4.64.2 – "βοὸς πλευρῇ" – In a rare singular usage, the Scythians scalp their first victims and scrape the brains off the skin with the "rib of a cow."

Antiphon - First surviving mathematical usage of the word to mean side of a triangle (from the 5th century B.C.E.).

Plato - Frequently uses πλευρά to refer to sides of polygons. In Timaeus 53d, he refers to the sides of a right triangle. In Theaetetus 148a, he uses it for the sides of a rectangle representing a factor of a number.

Archimedes - About Spheres and Cylinders - Uses πλευρά to refer to the generator of a cone. Technically, a cone is not a solid object like a sphere, but rather a surface. By intersecting the cone with a plane, the conic sections (circle, parabola, hyperbola) are formed. The generator of a cone is the line that is rotated around a vertical axis to produce the surface of a cone. This convention may or may not be followed by Apollonius of Perga, the greatest classical writer on the conic sections.

σύμμετρος

This is a rare word who's main meaning is mathematical. It means *commensurate.* If two segments, areas, or volumes can be expressed as multiples of the same base unit (μονάς), they are commensurate. Our word symmetric comes from this, but does not mean the same thing. Today, symmetric figures are identical to each other.

Our symmetric is closer to the way σύμμετρος was used outside of mathematics. In common parley, it meant *coincident with, keeping even with, or the same as* (Aeschylus).

Significant Uses:

Aeschylus – Libation Bearers: 230 – "σύμμετρον τὠμῷ κάρᾳ" – Orestes proves his identity to Electra by showing that the lock of hair he gave her is "the same as the one on his head."

Plato – Theaetetus: 147d – “μήκει οὐ σύμμετροι τῇ ποδιαίᾳ” – Here Plato refers to a proof by Theodorus which states that the sides of squares that have 3 ft^2 and 5 ft^2 as their respective areas are incommensurable with 1 foot. In other words, $\sqrt{3}$ and $\sqrt{5}$ are irrational.

Aristotle – Nicomachean Ethics: 1133b22 – “τοῦτο γὰρ πάντα ποιεῖ σύμμετρα” – He argues for a set prices and exchange rates. This would make all goods “commensurable” and thereby their relative worth would be firmly established.

Appendix Two: Annotated Bibliography

Allman, George J. Greek Geometry: Thales to Euclid. Dublin: Dublin UP, 1889.

This book was a relatively unknown scholarly effort, published by the author's school. It does not actually have a table of contents. It gives a detailed explanation of the advancements of Thales, Pythagoras, Archytas, Eudoxus, Menaechmus, Aristaeus, and Theaetetus.

Baron, Margaret E. "Greek Mathematics." History of Mathematics: Origins and Development of the Calculus 1. Great Britain: Open UP, 1974. 1-51.

This is the first chapter in a textbook on the history of calculus. It covers many of the same elements as this paper does. It is a very good textbook for a course. It has good emphasis on primary sources, and is even meant to be accompianied by an English translation of Euclid. In addition to the specifically calculus related sections of the chapter, Baron spends quite a bit of time on a general introduction to Greek math (basic proofs, counting methods, notations).This chapter serves as a wonderful first treatment of the subject of Greek calculus studies, and provides many avenues for further study.

Boyer, Carl B. The History of the Calculus and its Conceptual Development. New York, NY: Dover, 1949.

This is the first math history book I've read, which was written by a mathematician. It focused on the technical aspects of the development of calculus. The first Chapter is entitled “Conceptions in Antiquity”. It was very informative as a starting point for further research. It makes references to classical and modern works which can be easily followed to find out more. A very valuable resource.

Brunes, Tons. The Secrets of Ancient Geometry. Trans. Charles M. Napier. Vol I & II. Copenhagen, Sweeden: Rhodos, 1967.

These two volumes begin with information on the creation/discovery of mathematics, but spend most of their pages on math's application in architecture and elsewhere. The author claims that mathematics was somehow the fiber that held the ancient world together. He tries to find mystic connections and patterns from temple architecture to Platonic dialogues. The book does not deal with the intellectual development in which I am interested.

Capella, Martianus. De Nuptiis Philologiae et Mercurii. Ed. Adolf Dick. Stuttgart, Germany: B. G. Teubner, 1969.

This is a scholarly edition of Capella. All in Latin. The edition is well annotated and legible, and basically the only one printed in the last century.

Capella, Martianus. The Marriage of Philology and Mercury. Trans. E L. Burge, Richard Johnson, and William H. Stahl. New York: Columbia UP, 1977.

Books VI and VII of Capella are Geometry and Arithmetic. Book VI has Geometry personified give a speech, in which she claims to be the source of all things in the universe. She demands that Jove acknowledge that she was his creator. Geometry proceeds to give a detailed account of the Earth's geography. She describes its size and the positions of the continents. She talks about the celestial bodies; which stars can be seen in what continents, and how that proves the curvature of the Earth. Capella's discussion of the Earth's curvature is mostly sound, although a bit confused, and severely lacking in proof. However, when he begins to describe Eratosthanes' calculation of the Earth's circumference, he completely screws it up, showing his utter lack of mathematical understanding. In the final few pages of the book, Geometry begins to speak about her own subject, but basically, she refers the reader to Euclid.

The next book, Arithmetic is much more mathematically minded. It starts with a description of the mystical description of numbers 1 through 10, common in the classical world. This is mercifully brief, though, and most of the Arithmetic book is actually devoted to what we call Number Theory today. The goddess discusses divisibility, even and odd numbers and prime numbers in depth. Division is approached in an very interesting way. Capella's system of math does not allow for fractions in the same way we do. He discusses halves, thirds, and fourths, but never does he mention three fourths. He never proves any of his assertions. He will make a statement which requires mathematical proof, but simply follow it with one example. As a math major, it made me twitch.

Chase, Arnold B. The Rhind Mathematical Papyrus. Oberlin, Ohio: Mathematical Association of America, 1927.

This is an excellent edition of one of the most ancient mathematical texts around. The Rhind Papyrus dates back 19 centuries before Christ. It describes how the ancient Egyptians performed arithmetic, how they measured distances, and the geometrical knowledge used to construct the Pyramids. They had an interesting system to deal with fractions. They had no problem creating fractions with large denominators, but they could have no number but one in the numerator. They would use the hieratic script for the denominator (say, twenty-eight), but put a bar above it to indicate one twenty-eighth. So, to describe three fourths, they would use one half plus one quarter. This book shows one of the earliest well developed and codified mathematical systems at its height.

Danzig, Tobias. The Bequest of the Greeks. New York: Greenwood P, 1954.

This is a wonderfully written book about the methods and ideas that have actually survived and influenced modern mathematics. It is intended for the intellectual mathematician or the casually interested layman. The book describes the formation and uses of methods and formulae which are still used routinely today. It does not, however, give any useful information about the early foundations of calculus.

Euclid. Euclidis Elementa. Ed. J L. Heiberg. Leipzig: Teubner, 1883.

This is the current standard scholarly edition of Euclid in the original Greek. I used it via the Perseus project for most of my needs. However, Perseus does not have Heiberg's appendices, which I needed to consult about the __ proof. Since I accessed this text in book form, I list it here, though I don't list any of the other texts, which I accessed via Perseus or through Thomas's Loeb compendium.

Euclid. The Thirteen books of Euclid's Elements. Trans. Sir Thomas L. Heath. Vol. I-III. New York: Dover Publications, 1956.

Heath's translation is the agreed-upon standard for Euclid in English. It is the translation which Perseus has available. Thomas, who transated and compiled the Greek Math Loeb, was deeply indebted to Heath for consultation and assistance in that work.

Greek Mathematical Works. Trans. Ivor Thomas. Vol. I. Cambridge, Massachusetts: Harvard

UP, 1939.

Greek Mathematical Works. Trans. Ivor Thomas. Vol. II. Cambridge, Massachusetts:Harvard UP, 1939.

This contains much of Euclid's elements, as well as a vast group of other Greek sources for mathematical knowledge. It will be most useful as a primary source to refer to when I find references in modern works.

Heath, Thomas L. A Manual of Greek Mathematics. New York: Dover Publications, 1963.

This book is truly unique. It is basically all the practical mathematics the Greeks knew translated into modern notation. It is a grand summary of the solution methods the ancients used. It is a fantastic reference and very well organized.

Heath, Thomas L. Diophantus of Alexandria: A Study in the History of Greek Algebra. 2nd ed. New York: Dover Publications, Inc., 1964.

This is the first, and best English translation of Diophantus. Heath is a giant in the field of classical math history. This edition of Diophantus contains 120 pages of introductory material on Diophantus and his mathematical methods. The supplements also contain solutions by Fermat and Euler to questions Diophantus posed. All together, this is a marvelous scholarly work in the field of algebra.

Klein, Jacob. Greek Mathematical Thought and the Origin of Algebra. Trans. Eva Brann. Cambridge, MA: The M.I.T. P, 1969.

The book contains a good index of Greek mathematical terms, and a small bibliography of classical and German language sources. This is a complete and thorough analysis of the history of mathematical concepts relating to number. Discussion starts with Plato and his concept of arithmos and logistike, both types of number without a clear separation between the two (theoretical vs. calculation?). The discussion of number and it's meaning continues through to Rennaisance times with Des Cartes.

Lasserre, Francois. The Birth of Mathematics in the Age of Plato. Trans. Helen Mortimer. New York: Meridian Books, 1964.

This is actually a fairly general, basic book on the history of math. It focuses on concepts and definitions and contrasts the different schools of thought at the time of Plato. It does not go into depth in any area.

Neugebauer, O. The Exact Sciences in Antiquity. Providence: Brown University Press, 1957.

This book is a very comprehensive study of methods of computation in the ancient world. It covers Egypt, Babylon, Greece, and the Hellenistic world. It discusses methods useful in every discipleine from architecture and astronomy.

Smith, David E. Our Debt to Greece and Rome: Mathematics. New York: Cooper Square, 1963.

This book is a good overview of the major developments of Greek mathematics. It reviews the differences between arithmos and logistic number. It does discuss calculus, but only briefly and in very little detail.

Van der Waerden, B. L. Science Awakening. Trans. Arnold Dresden. New York: Oxford UP, 1961.

This is an attempt to document the beginnings of mathematical thought from Pythagoras and Thales, to Euclid. This book focuses on the earlier mathematicians, Thales and Pythagoras. We have no actual writings from these two, which makes scholarship on their work increasingly difficult. Waerden investigates the fairly recent scholarship on Egyptian and Babylonian math, and

the theorems attributed to Thales and Pythagoras, comparing and contrasting. He shows how they could have started from the pre-existing mathematical concepts and worked their way to what we identify as their contributions today.

Matthew Katsenes
Final Report
Wednesday, May 05, 2004

From Infinity to Irrationality and Back Again:
An investigation of the classical foundations of the calculus.

Abstract:

This paper is an introduction to Greek math, focusing on the development of calculus. It includes a brief overview of Greek mathematical history, and there is a fair amount about Euclid himself, since he is our primary source for knowledge of Greek math. Next, I begin to deal with the subject of calculus. They Pythagoreans were the first to move in that direction, and their discovery of irrational numbers was a crucial step. The Pythagorean theorem is the tool which first showed the existence of such numbers, and I analyze Euclid's proof. Then I analyze the proof of the irrationality of the square root of two, which is found interpolated in Euclid.

At the end of the paper, I analyze two more mathematical concepts which spurred on the development of calculus. First, there are the Paradoxes of Zeno. These each deal with infinity and the infinitesimal, and provoked much thought and study which eventually led to calculus. The second concept I touch on is the squaring of the circle. The problem of comparing the areas of round objects with rectilinear objects was one which puzzled the ancients because they did not understand that π is an irrational number, which makes it impossible to compare the area of a square and circle using their methods. Eudoxus came the closest with his method of exhaustion, which is strikingly similar to modern integral calculus.

The last point about Eudoxus inspired me to close with a few words on the difficulty of finding the true creator of any proof or concept. So few texts remain, and

those that do are full of contradictory remarks, good and bad proofs, and unmeasurable biases.

After the paper, I have created a glossary of Greek mathematical terms. The glossary includes some of the most interesting and challenging terms I encountered in my study.

Method:

I began this project last fall with the vague intention of writing something about classical mathematics to tie my two majors together. The first source I looked at was Martianus Capella because he was the only Latin mathematical author I could find. His material turned out to be of extremely poor quality mathematically. At this point I realized I would have to turn to Greek sources.

The first Greek math source I found was the Loeb collection of mathematical authors. This turned out to be an immensely useful source throughout my project. At this point I began to turn to general surveys of math history and went to ask Dr. Dwyer for ideas. He loaned me a book on the development of calculus which had an excellent chapter on classical developments (Boyer). This became the focus of my paper.

I thought back to the proofs I had seen in mathematics classes, and went to my real analysis book to see how that author began. Everywhere I looked, I realized that the mathematical foundations of calculus lie in the continuity of the real number line. It became a quest for me to find the proof of the discontinuity of the numbering system of the Greeks (the rational numbers). I had seen it several times in a modern format, and knew it existed in classical times. Eventually, it took a trip to the University of Illinois and a hunt through their library to find it, but it was an exciting read once I did find it.

These are the main points in the development of this paper. There was much more research going on during all this. Finding calculus as my topic came only after reading several books focusing on other aspects of math, including one interesting one on the occult nature of math.

What I learned:

This project was an extensive learning process. It began with simply learning where to look for resources on math history, classics, and everything in between. I requested countless inter-library loans before I learned to properly skim the summary for my particular interests.

From these basic research principles, I moved on into the world of math history. I learned just how much of the scholarship in this field is not done in the English language. I found many references to seminal German works in the field. This has influenced me enough that in graduate school, I will be learning German because I know how useful it is in the math world.

Once I had a topic chosen and focused research under way, I began to actually formulate my paper. This was a chore unlike any I'd ever undertaken. Intimidated by the entire process, I put it off for quite a while, but when I did start to write my Eta Sigma Phi draft, I realized that I really had done a lot of research. The paper flowed naturally as I poured out all the knowledge I had gained in my research. The task that seemed so daunting was reduced to something manageable and I fought through it.

I now would be much more comfortable taking on a project of this magnitude again. I have learned a great deal about the general processes of research and writing, which I know will benefit me in graduate school.

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Evaluation of Matthew Katsenes' Project on an Investigation into the Classical Foundations of Calculus

The bibliography demonstrates access to a variety of print resources on the history of mathematics. One wonders whether additional materials might also be available on the internet. The annotations are detailed and informative.

The glossary of important Greek mathematical terms is excellent and is, perhaps, the best and most original portion of this paper. Even a non-mathematician benefits from reading the way ancient mathematicians applied Greek words to mathematical concepts. Occasionally some additional analysis of non-mathematical materials would have enhanced this discussion; for example, more commentary like that on Euripides's use of *arithmos* at *Ion* 1014 would have been helpful. Such additions would be an appropriate follow-up for the next stage of this study, were it to be undertaken.

An appendix providing information on the major mathematicians and mathematical works of antiquity would have been an appropriate addition to this paper.

While the paper is generally well-written and demonstrates excellent use of the researched materials, it contains a variety of stylistic flaws and typos which would undoubtedly have been corrected in a revised version. Occasionally Greek characters appear only as boxes, (e.g., pg. 19).

The paper itself offers a sweeping overview and history of ancient mathematics with a focus on several important ancient proofs. Unfortunately, the paper usually does not go much beyond presentation of the proofs and offers little analysis from a modern mathematical point of view. For example, the concluding paragraph on Zeno's paradoxes states only that they "have been the subject of much discussion in the mathematical world to this day." The layperson would really like to know what sorts of discussions these would be.

This study would make an excellent submission to the Eta Sigma Phi panel for CAMWS Southern Section 2004. Perhaps a combination of one or two proofs with appropriate material from the appendix would be effective.

I think you should be proud of the way you combined your two academic interests in this project, Matthew. You might even consider showing this work to your professors at the University of Iowa to see what they think of it. Based upon the comments I have provided above, I think your paper has earned a grade of A-. I would also evaluate your whole project, including research and writing, as an A-.

While I continue to wish that you had been able to continue your study of the Classics into graduate school, I have certainly enjoyed working with you for the past two years and wish you all the best as you pursue your graduate studies in mathematics at the University of Iowa.

Si vales, valeo.

Thomas J. Sienkewicz
Minnie Billings Capron Professor of Classics

Call for Papers

for presentation at the
Classical Association of the Middle West and South
Southern Section Meeting
November 4-6, 2004
in Winston-Salem, North Carolina
at the invitation of the Wake Forest University
and in cooperation with
the University of North Carolina-Greensboro and Davidson College

At the meeting of the Southern Section of CAMWS, Eta Sigma Phi will sponsor a panel of papers presented by *undergraduate* members of Eta Sigma Phi. Members who will be undergraduates in the fall (or who graduated in the spring of 2004) are invited to submit papers for consideration, and five or six papers will be selected for presentation.

The papers will be judged anonymously, and the students whose papers are selected for reading will receive $100 each to help cover expenses of attending the meeting. They will also be given a one-year membership in CAMWS. Before submitting a paper, each student should ensure that he or she will be able to obtain the additional funds—either personally or through the institution, department, or chapter—to attend the meeting.

Requirements:

1. **Papers should deal with some aspect of classical civilization or language. (Papers written for classes are acceptable.)**
2. **Papers should be typed, double-spaced, and no longer than 15 minutes in length, or 20 minutes if audio-visuals are part of the presentation.**
3. **The names of the authors should not be on the papers.**
4. **Each submission should contain a cover sheet with the author's name, address, phone number, e-mail address, chapter, and institution. Those who will not be at their institutions in June should also include summer information.**

Deadline for receipt of papers: June 1, 2004

Send your papers to:
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